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Quantum mechanics of non-linear systems

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Abstract. A generalisation of Weyl's correspondence rule and the associated Moyal bracket for systems with non-linear phase spaces that admit symplectic transitive group actions is presented. This provides a simple account with explicit constructions of some of the general results of Bayen *et al.*

1. Introduction

The standard Dirac quantisation algorithm of replacing Poisson brackets with operator commutators does not provide a completely satisfactory connection between the classical and quantum theories. Although the intuitive ideas underlying this relation are clear, the whole procedure cannot be followed through. Van Hove's theorem (see, e.g., Chernoff 1981) proves the existence of mathematical obstructions to the construction of a full quantisation map. In 1949, Moyal introduced a new bracket for functions on the classical phase space that replaces the Poisson one in the quantisation procedure. This bracket is closely related to Weyl's correspondence rule between classical and quantum observables (Weyl 1928). The new Lie algebra associated with this bracket is a deformation of the Poisson Lie algebra which is recovered in the contraction limit $\hbar \to 0$.

Later Bayen *et al* (1978) presented an extensive study of deformations of the algebras associated with general phase spaces and argued that it is possible to understand quantisation as a deformation of the classical Poisson Lie algebra of observables. This involved a general mathematical study of such deformations together with applications to various quantum mechanical systems. Since then, the same group of authors and others have developed these ideas further (see, e.g., Arnal *et al* 1983, Cahen and Gutts 1982, Fronsdal 1978, Lichnerowicz 1983) (for an extensive list of references, see NATO (1986)).

In this paper we study the generalisation of Weyl's correspondence rule and the Moyal bracket to systems with non-linear phase spaces that admit symplectic transitive group actions. Our approach follows simple arguments with emphasis on the correspondence rules providing a simple account with explicit constructions of the general results of Bayen *et al* for this class of systems. We review briefly (§ 2) a quantisation scheme for the kinematics of these systems which provides globally well defined basic observables through a symplectic transitive G-group action. Within this framework a correspondence rule together with an associated deformation of the classical Poisson algebra is constructed (§ 3). This coincides with the general $L(\mathrm{G})$-invariant $*$ product of Bayen

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et al (1978). In our work, the choice of an invariant subalgebra of the Poisson algebra of all physical observables is used first to construct correspondence rules. The new algebra is then constructed from these rules rather than through general mathematical considerations of deformations of the Poisson algebra. In § 4, a discussion of some examples is presented.

2. Selection of basic observables

Consider a classical physical system whose phase space is the symplectic manifold S. Dynamical evolution manifests itself as the flow lines of the Hamiltonian vector field ξ_H, defined in terms of the Hamiltonian function $H \in C^\infty(S, \mathbb{R})$ as

$$i_{\xi_H}\omega := \mathrm{d}H \tag{2.1}$$

where ω is the symplectic 2-form on S and $C^\infty(S, \mathbb{R})$ is the space of infinitely differentiable functions from S into $\mathbb{R}$. Here, $i_{\xi_H}\omega$ denotes the interior product of the 2-form ω with the vector field ξ_H (and hence is a 1-form). The map from $C^\infty(S, \mathbb{R})$ into HVF(S), the space of Hamiltonian vector fields on S defined by (2.1), is a homomorphism with kernel the real numbers $\mathbb{R}$, as any two Hamiltonian functions that differ by a constant $c \in \mathbb{R}$ provide the same element in HVF(S). The Poisson bracket of two functions f, $g \in C^\infty(S, \mathbb{R})$ is defined to be

$$\{f, g\}_{\mathrm{PB}} := \omega(\xi_f, \xi_g) \tag{2.2}$$

(see Abraham and Marsden (1978) for more details of symplectic geometry).

Darboux's theorem asserts the existence of a local coordinate system (q^k, p_j) on S, $k, j = 1, 2, \ldots, \frac{1}{2}\dim S$, called canonical coordinates, in which the symplectic form becomes $\omega = \mathrm{d}q^k \wedge \mathrm{d}p_k$ and hence the Poisson bracket takes the (familiar) form

$$\{f, g\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q^k}\frac{\partial f}{\partial p_k}. \tag{2.3}$$

The space $C^\infty(S, \mathbb{R})$ is endowed with two different mathematical structures: an associative Abelian algebra under the usual dot product of functions and a Lie algebra under the Poisson bracket.

In the canonical quantisation of a classical system one traditionally attempts to impose the Heisenberg commutation relations,

$$[\hat{q}^k, \hat{p}_j] = \mathrm{i}\hbar\delta^k_j\mathbb{1} \tag{2.4}$$

between the basic observables. (In (2.4) and from now on $\mathrm{i} = \sqrt{-1}$.) However, if S is not a linear space, the canonical coordinates q^k, p_j are not globally defined and the commutation relations (2.4) may not be appropriate.

Recently, a method has been proposed (Isham 1984, Isham and Kakas 1984) that selects a set of globally defined basic observables in a group theoretical manner. One considers phase spaces that admit a connected Lie group G acting symplectically, transitively and almost effectively†. Then the map

$$\gamma : L(\mathrm{G}) \rightarrow \mathrm{HVF}(S) \tag{2.5}$$

† For linear phase spaces, the Weyl–Heisenberg group with commutation relations (2.4) acts in this way by translations.

that attaches a HVF γ^A to each element $A \in L(\mathrm{G})$ of the Lie algebra of the group G, via

$$\gamma_s^A(f) := \frac{\mathrm{d}f}{\mathrm{d}t}(\exp(-tA)s)_{|t=0} \tag{2.6}$$

where $s \in S$ and $f \in C^\infty(S, \mathbb{R})$, is an isomorphism into the set of Hamiltonian vector fields on S. This is ensured by the action of G on S considered above. We now associate the preferred class of physical observables, $P_A \in C^\infty(S, \mathbb{R})$, with $L(\mathrm{G})$, defined by the following equation:

$$\gamma^A = -\xi_{P_A}. \tag{2.7}$$

This is summarised in the diagram

$$\begin{array}{c} 0 \to \mathbb{R} \hookrightarrow C^\infty(S, \mathbb{R}) \to \mathrm{HVF}(S) \to 0. \\ {}^{P}\nwarrow \quad \uparrow{}^{\gamma} \\ L(\mathrm{G}) \end{array} \tag{2.8}$$

In (2.8), the sequence of maps

$$0 \to \mathbb{R} \to C^\infty(S, \mathbb{R}) \to \mathrm{HVF}(S) \to 0$$

is an exact sequence†; $\mathbb{R} \to C^\infty(S, \mathbb{R})$ is the natural inclusion of $\mathbb{R}$ (the constant functions) in $C^\infty(S, \mathbb{R})$ and $\mathbb{R}$ also coincides with the kernel of the homomorphism $C^\infty(S, \mathbb{R}) \to$ HVF(S) defined by (2.1). The induced map $P: L(\mathrm{G}) \to C^\infty(S, \mathbb{R})$ (i.e. P: $A \rightsquigarrow P_A$) is called the momentum map. Furthermore, we demand that P be linear and a Lie algebra isomorphism:

$$\{P_A, P_B\}_{\mathrm{PB}} = P_{[A,B]} \qquad A, B \in L(\mathrm{G}). \tag{2.9}$$

In general this may not be possible, since $\{P_A, P_B\}_{\mathrm{PB}} - P_{[A,B]} \neq 0$. Indeed, using the maps introduced above we note that $-\xi_{P_{[A,B]}} = \gamma^{[A,B]} = [\gamma^A, \gamma^B] = [\xi_{P_A}, \xi_{P_B}] = -\xi_{\{P_A, P_B\}_{\mathrm{PB}}}$ and hence (since the kernel of (2.1) is $\mathbb{R}$)

$$\{P_A, P_B\}_{\mathrm{PB}} - P_{[A,B]} = z(A, B) \in \mathbb{R} \tag{2.10}$$

holds in general. $z: L(\mathrm{G}) \times L(\mathrm{G}) \to \mathbb{R}$ is a real valued 2-cocycle of the Lie algebra cohomology. As such, it satisfies the properties

(i) $z(A, B) = -z(B, A)$

(ii) $z(A, [B, C]) + z(B, [C, A]) + z(C, [A, B]) = 0.$

A 2-cocycle is trivial if it has the form $z(A, B) = \langle d, [A, B] \rangle$ for some $d \in L^*(\mathrm{G})$, the dual of the Lie algebra of G ($\langle\ \rangle$ denotes the usual pairing between $L^*(\mathrm{G})$ and $L(\mathrm{G})$). In this case we define

$$P'_A = P_A + \langle d, A \rangle \tag{2.11}$$

which satisfy (2.9). If z is not trivial, then we extend centrally $L(\mathrm{G})$ with the aid of $\mathbb{R}$ and obtain $L(\mathrm{G}') = L(\mathrm{G}) \oplus \mathbb{R}$ in order to achieve the desired Lie algebra isomorphism. The group G (or G′) will be called the canonical group and its Lie algebra generators $\{P_A\}$ provide a preferred class of globally defined observables for quantisation on S. The study of the unitary irreducible representations (UIR) of the canonical group G

† Recall that a sequence of homomorphisms is exact if, for each pair of consecutive maps, the image of the first is equal to the kernel of the second. Here '0' denotes zero ($0 \in \mathbb{R}$).

(denoting from now on G or G′) provides the quantum Hilbert space $\mathscr{H}$ and representations of the basic observables $\{P_A, A \in L(G)\}$ with self-adjoint operators $\hat{A}$ on $\mathscr{H}$. For systems with linear phase spaces, the Heisenberg-Weyl group (G_W) is singled out with (2.4) as commutation relations, but for non-linear theories more general canonical groups result. We present the examples $S = \mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}_+$ in § 4, although the interested reader may find more details and examples of the general construction for selecting the set of basic observables in Isham (1984).

3. General construction

3.1. Correspondence rule

The study of quantum dynamics requires the construction of a linear map ϑ (correspondence rule) that associates an unambiguous self-adjoint (or at least Hermitian) operator $\vartheta(f)$ on $\mathscr{H}$ with the classical physical observables $f \in C^\infty(S, \mathbb{R})$. It has to satisfy $\vartheta(P_{A_l}) = \hat{A}_l$ (in any particular UIR of G) for all the basic observables, which in turn implies

$$\vartheta(\{P_{A_l}, P_{A_j}\}_{PB}) = -(i/\hbar)[\hat{A}_l, \hat{A}_j]. \tag{3.1}$$

Because of Van Hove's theorem (Chernoff 1981), such a map cannot be a full quantisation (at least for theories with linear phase spaces), i.e. it is not possible to satisfy (3.1) for an arbitrary pair of physical observables $f_1, f_2 \in C^\infty(S, \mathbb{R})$. To overcome this, we construct a new $\hbar$-dependent classical algebra with the property that it tends to the usual one as $\hbar \to 0$, i.e. it is a deformation of the Poisson algebra.

The global group theoretical framework for handling the kinematical aspects of quantisation described in the previous section affords a natural construction for a correspondence rule†. The kinematical observables of the classical theory are in general required to have the property that every other physical observable can be expressed in terms of these. (This then holds in the quantum theory by requiring that the representation of the canonical group is irreducible.)

The transitivity of the canonical group action on S ensures that every function $f \in C^\infty(S, \mathbb{R})$ can be locally expressed as a function of the basic observables $\{P_{A_k}\}$. This is due to the fact that the G action enables us to locally embed the phase space S in the dual of the Lie algebra of G, $L^*(G)$, as an orbit of the coadjoint action of G on $L^*(G)$ (Kirillov orbit). Recall that the coadjoint action ad_g^* of G on $L^*(G)$ is defined by

$$\langle \mathrm{ad}_g^* \, d, A\rangle = \langle d, \mathrm{ad}_g A\rangle \qquad d \in L^*(G), A \in L(G), g \in G$$

where ad_g is the adjoint action of G on $L(G)$. It is known that the orbits of the coadjoint action are symplectic manifolds (Kirillov 1976a, b). The immersion $J: S \to L^*(G)$, known as Souriau's J-momentum map (Souriau 1970), is defined in terms of the momentum map P via

$$\langle J(s), A_k\rangle = P_{A_k}(s) \qquad s \in S. \tag{3.2}$$

† In trying to extend the Weyl correspondence to theories with a general homogeneous phase space S ($= G/G_0$), one might conjecture that harmonic analysis on G/G_0 will play a major role. However, this can give rise to correspondence rules only for special cases and hence cannot provide a complete generalisation.

To make more precise the connection with the orbits of the coadjoint action, we consider the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{J} & L^*(G) \\ {\scriptstyle l_g}\downarrow & & \downarrow{\scriptstyle \tau_g} \\ S & \xrightarrow{J} & L^*(G) \end{array} \qquad l_g(s) := gs \qquad s \in S$$

where $\tau_g = \mathrm{ad}^*_{g^{-1}} + z(g)$. J is called ad^*_G equivariant if $z(g) = 0$. (In general, the obstructions to ad^*_G equivariance are provided by the 1-cocycles $z: G \to L^*(G)$ of the group cohomology; their 'infinitesimal' version is given by the Lie algebra 2-cocycles $z(A, B)$ of § 2.) Here we are considering ad^*_G equivariance for J and so from the above commutative diagram we have that $J(gs) = \mathrm{ad}^*_{g^{-1}}(J(s))$, i.e. J maps S onto an orbit of the ad* action of G on $L^*(G)$. Further technical details can be found in Abraham and Marsden (1978).

We will consider here only those physical observables f_0 which can be expressed globally as a function of the basic observables $\{P_{A_l}\}$. Hence there exists a function F_{f_0} in $C^\infty(\mathbb{R}^n, \mathbb{R})$ such that, for each $s \in S$,

$$f_0(s) = F_{f_0}(P_{A_1}(s), \ldots, P_{A_n}(s)) \tag{3.3}$$

i.e.

$$f_0 = F_{f_0} \cdot F \tag{3.4}$$

where $F: S \to \mathbb{R}^n$ satisfies

$$F(s) = (\langle J(s), A_1 \rangle, \ldots, \langle J(s), A_n \rangle). \tag{3.5}$$

Consequently, an observable f_0 satisfying equation (3.4) can be uniquely associated with a function $f: L^*(G) \to \mathbb{R}$ defined by

$$f(x) := F_{f_0}(x_1, \ldots, x_n) \tag{3.6}$$

where $x_l := \langle x, A_l \rangle$ are the coordinate functions on $L^*(G)$. The functions f_0 and f are related by

$$f_0(s) = f \cdot J(s). \tag{3.7}$$

Having associated the classical observables f_0 on S with functions on the linear space $L^*(G) \simeq \mathbb{R}^n$, we can apply Fourier analysis to obtain

$$f(x) = \frac{1}{\sqrt{(2\pi)^n}} \int_{L(G)} \tilde{f}(\alpha) \exp(\mathrm{i}\langle x, \alpha \rangle)\, \mathrm{d}^n\alpha \tag{3.8}$$

where

$$\tilde{f}(\alpha) = \frac{1}{\sqrt{(2\pi)^n}} \int_{L^*(G)} f(x) \exp(-\mathrm{i}\langle x, \alpha \rangle)\, \mathrm{d}^n x. \tag{3.9}$$

For any point $x \in L^*(G)$ such that $x = J(s)$ for some $x \in S$, we have from (3.8), using equation (3.4),

$$\begin{aligned} f(J(s)) &= \frac{1}{\sqrt{(2\pi)^n}} \int_{L(G)} \tilde{f}(\alpha) \exp(\mathrm{i}\langle J(s), \alpha \rangle)\, \mathrm{d}^n\alpha \\ &= \frac{1}{\sqrt{(2\pi)^n}} \int_{L(G)} \tilde{f}(\alpha) \exp\left(\mathrm{i} \sum_{l=1}^{n} \alpha^l P_{A_l}(s)\right) \mathrm{d}^n\alpha \end{aligned}$$

where $\alpha=\Sigma_{k=1}^{n}\alpha^k A_k$ (and so by (3.2), $\langle J(s),\alpha\rangle=\Sigma_{k=1}^{n}\alpha^k\langle J(s),A_k\rangle=\Sigma_{k=1}^{n}\alpha^k P_{A_k}$). Hence, from equation (3.7),

$$f_0(s)=\frac{1}{\sqrt{(2\pi)^n}}\int_{L(G)}\tilde{f}(\alpha)\exp\left(\mathrm{i}\sum_{l=1}^{n}\alpha^l P_{A_l}(s)\right)\mathrm{d}^n\alpha. \tag{3.10}$$

Equation (3.10) has the desirable feature that the basic observables appear in a linear way and hence we can define the quantum operator $\hat{F}_0$ corresponding to f_0 by

$$\hat{F}_0(\hat{A}_1,\ldots,\hat{A}_n):=\frac{1}{\sqrt{(2\pi)^n}}\int_{L(G)}\tilde{f}(\alpha)\exp\left(\mathrm{i}\sum_{l=1}^{n}\alpha^l\hat{A}_l\right)\mathrm{d}^n\alpha. \tag{3.11}$$

This is a generalisation of Weyl's correspondence rule for systems with non-linear phase space.

At this stage, we would like to clarify a point that has not been mentioned so far. In general, the centre of the universal enveloping algebra of a Lie algebra $L(\mathrm{G})$ is not trivial, which implies the existence of Casimirs. They are constant on the Kirillov orbits and they give rise to Casimir functions C over the whole of $L^*(\mathrm{G})$, i.e. $\{C,f\}=0$ for all $f\in C^\infty(L^*(\mathrm{G}),\mathbb{R})$, where the Poisson bracket on $C^\infty(L^*(\mathrm{G}),\mathbb{R})$ is defined via (Kirillov 1976a, b)

$$\{f_1,f_2\}:=\sum_{l,j,k=1}^{n}C_k^{lj}x^k\frac{\partial f_1}{\partial x^l}\frac{\partial f_2}{\partial x^j}$$

C_k^{lj} being the structure constants of $L(\mathrm{G})$. Upon quantisation, they are proportional to the identity operator, and so the correspondence rule $f_0\to\hat{F}_0$, described via (3.11), is well defined despite the existence of Casimirs.

The operator $\hat{F}_0$ in (3.11) is defined in the weak sense, i.e. in terms of the matrix elements

$$\int\mathrm{d}^n\alpha\int\mathrm{d}^n x f(x)\exp(-\mathrm{i}\langle x,\alpha\rangle)\left\langle\psi_1,\exp\left(\mathrm{i}\sum_{l=1}^{n}\alpha^l\hat{A}_l\right)\psi_2\right\rangle$$

where ψ_1,ψ_2 are elements of the Hilbert space $\mathscr{H}$ of the irreducible representation of G that one chooses for quantisation of the kinematics. There are no obvious conditions on f that ensure that these integrals will converge but in our framework these, and hence the operators $\hat{F}_0$, exist for a large class of observables. Specifically, whenever f_0 is a polynomial of the $\{P_{A_l}\}$, since $\tilde{f}(\alpha)$ is a distribution of finite support these integrals clearly exist in the distributional sense for all ψ in the Garding domain of the UIR of G. Under the uniform convergence topology of $C^\infty(S,\mathbb{R})$, polynomial functions are dense in the space of functions of compact support on $\mathbb{R}^n$. Hence, for any observable f_0 where f has compact support, there exists a sequence of polynomials (f_m) that tends to f. The corresponding sequence $\hat{F}_m$ of operators converges as $m\to\infty$ due to the boundedness of the operator $\exp(\mathrm{i}\,\Sigma_{l=1}^{n}\alpha^l\hat{A}_l)$. The operator $\hat{F}_0$ is then defined to be this limit†. (From now on, we drop the 0 subscript for convenience.)

The correspondence rule ϑ described by (3.11) provides an unambiguous one-to-one correspondence between classical and Hermitian quantum observables $\hat{F}\subset\hat{F}^+$‡.

† We note that the theory of pseudodifferential operators (Hörmander 1979) provides an appropriate mathematical framework for (3.11) in the case of linear phase spaces and the Weyl group.

‡ To the best of our knowledge, there are no general conditions on f to ensure that $\hat{F}$ is self-adjoint. This problem is under investigation.

Moreover, it is a linear map with the required property $\vartheta(p_{A_l}) = \hat{A}_l$. Note that if $\hat{U}(\mathrm{g})$ is a UIR of the canonical group G, then the following intertwining relation holds:

$$\hat{U}(\mathrm{g})\hat{F}(A_1, \ldots, A_n)\hat{U}^{-1}(\mathrm{g}) = \hat{F}(\hat{P}_{\mathrm{ad_g} A_1}, \ldots, \hat{P}_{\mathrm{ad_g} A_n}). \tag{3.12}$$

The proof follows from the linearity of the momentum map P. Hence the correspondence rule (3.11) is a basis-independent construction as $A_l \to \mathrm{g}A_l\mathrm{g}^{-1}$ results in a unitary equivalent operator to $\hat{F}(\hat{A}_l)$.

3.2. Generalised structures

Classically, $C^\infty(S, \mathbb{R})$ is endowed with an Abelian associative algebra structure under the usual dot product of functions and with a Lie algebra structure under the Poisson bracket. Their precise relation with the corresponding quantum algebras is the essence of the present section. The whole idea here is based on the fact that, given a non-commutative associative algebra structure on the space of C^∞ functions over a manifold, a Lie algebra structure is induced by defining the Lie bracket to be the commutator with respect to the non-commutative product. This is a very important ingredient in investigating the relation between the classical and quantum algebraic structures, as the desired $\hbar$ deformation of the classical Poisson bracket may be induced by deforming the usual product of functions over the phase space to a non-commutative (but still associative) product. In particular, our goal would be to first introduce a non-Abelian associative algebra structure over the space of classical observables (isomorphic to the quantum one under the product of operators) and then induce a Lie algebra structure (isomorphic by construction to the quantum one under $-(\mathrm{i}/\hbar)[\ ,\ ]$), which admits a $\hbar$ expansion around the classical Poisson bracket algebra. More precisely, we have the following.

Proposition 3.1. The quantum algebra of operators under the dot product is isomorphic to the algebra of their corresponding classical functions (cf (3.11)) under $f_1 \circ f_2$ defined by

$$(f_1 \circ f_2)(P_{A_1}(s), \ldots, P_{A_n}(s)) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n\alpha\, \mathrm{d}^n\beta\, \tilde{f}_1(\alpha)\tilde{f}_2(\beta) \times \exp\left(\mathrm{i} \sum_{l=1}^{n} ((\alpha_1, \ldots, \alpha_n) \circ' (\beta_1, \ldots, \beta_n))_l P_{A_l}(s)\right) \tag{3.13}$$

where the $\circ'$ product denotes the G-group product law in the $(\alpha_1, \ldots, \alpha_n)$ parametrisation.

Proof. Consider the product $\hat{F}_1 \cdot \hat{F}_2$ of two quantum operators which can be expressed via (3.11) as

$$\hat{F}_1 \cdot \hat{F}_2 = \frac{1}{(2\pi)^n} \int \mathrm{d}^n\alpha\, \mathrm{d}^n\beta\, \tilde{f}_1(\alpha)\tilde{f}_2(\beta)\hat{U}(\alpha)\hat{U}(\beta) \tag{3.14}$$

where

$$\hat{U}(\alpha_1, \ldots, \alpha_n) := \exp\left(\mathrm{i} \sum_{l=1}^{n} \alpha^l \hat{A}_l\right) \tag{3.15}$$

is a UIR of the canonical group G. Let us introduce

$$(\tilde{f}_1^* \tau \tilde{f}_2)(\gamma) := \frac{1}{\sqrt{(2\pi)^n}} \int \mathrm{d}^n\alpha \frac{\Gamma(\alpha^{-1} \circ' \gamma)}{\Gamma(\gamma)} \tilde{f}_1(\alpha)\tilde{f}_2(\alpha^{-1} \circ' \gamma) \tag{3.16}$$

where $\alpha \circ' \beta \equiv (\alpha_1, \ldots, \alpha_n) \circ' (\beta_1, \ldots, \beta_n) := \gamma \equiv (\gamma_1, \ldots, \gamma_n)$ denotes the group law product in the $(\alpha_1, \ldots, \alpha_n)$ parametrisation (3.15), α^{-1} denotes the inverse with respect to $\circ'$ in the same parametrisation and μ is the left Haar measure on G,

$$\mathrm{d}^n\alpha \equiv \mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_n = \Gamma(\alpha)\, \mathrm{d}\mu(\alpha)$$

(the factor $\Gamma(\alpha)$ is defined in such a way that it compensates between $\mathrm{d}^n\alpha$ and $\mathrm{d}\mu(\alpha)$). Then $\hat{U}(\alpha)\hat{U}(\beta) = \hat{U}(\alpha \circ' \beta) = \hat{U}(\gamma)$ and, according to our notation, we have $\beta = \alpha^{-1} \circ' \gamma$. Consequently, using the left invariance of μ, we obtain

$$\begin{aligned}\mathrm{d}^n\beta \tilde{f}_2(\beta) &= \mathrm{d}^n(\alpha^{-1} \circ' \gamma)\tilde{f}_2(\alpha^{-1} \circ' \gamma) = \mathrm{d}\mu(\alpha^{-1} \circ' \gamma)\Gamma(\alpha^{-1} \circ' \gamma)\tilde{f}(\alpha^{-1} \circ' \gamma)\\ &= \mathrm{d}\mu(\gamma)\Gamma(\alpha^{-1} \circ' \gamma)\tilde{f}(\alpha^{-1} \circ \gamma) = \mathrm{d}^n\gamma \frac{\Gamma(\alpha^{-1} \circ' \gamma)}{\Gamma(\gamma)} \tilde{f}(\alpha^{-1} \circ' \gamma).\end{aligned}$$

So (3.14) takes the form

$$\hat{F}_1 \cdot \hat{F}_2 = \frac{1}{\sqrt{(2\pi)^n}} \int \mathrm{d}^n\gamma (\tilde{f}_1^* \tau \tilde{f}_2)(\gamma) \hat{U}(\gamma) \tag{3.17}$$

and hence, in view of the correspondence rule (3.11), the function $f_1 \circ f_2$ defined by

$$(f_1 \circ f_2)(P_{A_1}(s), \ldots, P_{A_n}(s)) := \frac{1}{\sqrt{(2\pi)^n}} \int \mathrm{d}^n\gamma (\tilde{f}_1^* \tau \tilde{f}_2)(\gamma) \exp\left(\mathrm{i} \sum_{k=1}^{n} \gamma_k P_{A_k}(s)\right) \tag{3.18}$$

has $\hat{F}_1 \cdot \hat{F}_2$ as its corresponding quantum operator. Combination of (3.16) and (3.18) and the use of the left invariance of the measure μ concludes the proof and simultaneously provides us with the definition of the $\circ$ product over the space of classical observables. The expression (3.13) makes manifest its close relation with the $\circ'$ product law of the canonical group: the non-Abelian nature of the canonical group is responsible for the non-commutativity of the $\circ$ product.

An immediate consequence of (3.13) is the following corollary.

Corollary 3.2. The $\circ$ product is associative and non-Abelian.

This product is a generalisation of the Moyal product (Moyal 1949). We will now use proposition 3.1 to construct a deformation of the Poisson Lie algebra. We introduce $\hbar$ in the commutation relations of $L(\mathrm{G})$ so that the following conditions are satisfied:

(i) $L(\mathrm{G}^\hbar)$ is isomorphic to $L(\mathrm{G}^{\hbar=1}) = L(\mathrm{G})$

(ii) $L(\mathrm{G}^\hbar)$ becomes Abelian as $\hbar \to 0$. (3.19)

For the canonical groups G considered by Isham, (3.19) can be imposed consistently (Isham 1984). These conditions provide the generalisation of the role that $\hbar$ plays in the Heisenberg-Weyl commutation relations to more general canonical groups. Consequently, the G-group product law $\alpha \circ' \beta$ admits a (Taylor) $\hbar$ expansion:

$$(\alpha \circ' \beta)_l = (\alpha + \beta)_l + \hbar \frac{\mathrm{d}(\alpha \circ' \beta)_l}{\mathrm{d}\hbar}\bigg|_{\hbar=0} + \frac{\hbar^2}{2} \frac{\mathrm{d}^2(\alpha \circ' \beta)_l}{\mathrm{d}\hbar^2}\bigg|_{\hbar=0} + \ldots \tag{3.20}$$

and so

$$\exp\left(\mathrm{i}\sum_{l=1}^{n}(\alpha\circ'\beta)_l P_{A_l}(s)\right)=\exp\left(\mathrm{i}\sum_{l=1}^{n}(\alpha+\beta)_l P_{A_l}(s)\right)\left[1+\mathrm{i}\hbar\sum_{k=1}^{n}\left.\frac{\mathrm{d}(\alpha\circ'\beta)_k}{\mathrm{d}\hbar}\right|_{\hbar=0}P_{A_k}(s)\right.$$
$$+\mathrm{i}\frac{\hbar^2}{2}\sum_{k=1}^{n}\left.\frac{\mathrm{d}^2(\alpha\circ'\beta)_k}{\mathrm{d}\hbar^2}\right|_{\hbar=0}P_{A_k}(s)$$
$$\left.-\frac{\hbar^2}{2}\left(\sum_{k=1}^{n}\left.\frac{\mathrm{d}(\alpha\circ'\beta)_k}{\mathrm{d}\hbar}\right|_{\hbar=0}P_{A_k}(s)\right)^2+\cdots\right]. \tag{3.21}$$

Therefore the $\circ$ product admits a $\hbar$ expansion as well:

$$(f_1\circ f_2)(P_{A_1}(s),\ldots,P_{A_n}(s))=(f_1\cdot f_2)(P_{A_1}(s),\ldots,P_{A_n}(s))$$
$$+\frac{\mathrm{i}\hbar}{(2\pi)^n}\int \mathrm{d}^n\alpha\,\mathrm{d}^n\beta\,\tilde{f}_1(\alpha)\tilde{f}_2(\beta)\sum_{k=1}^{n}\left.\frac{\mathrm{d}(\alpha\circ'\beta)_k}{\mathrm{d}\hbar}\right|_{\hbar=0}P_{A_k}(s)$$
$$\times\exp\left(\mathrm{i}\sum_{l=1}^{n}(\alpha+\beta)_l P_{A_l}(s)\right)+\ldots \tag{3.22}$$

thus being a non-trivial deformation of the usual dot product of functions

$$f_1\circ f_2=f_1\cdot f_2+\hbar K_1(f_1,f_2)+\hbar^2 K_2(f_1,f_2)+\ldots\,. \tag{3.23}$$

The following follows from the associativity of the $\circ$ product.

Corollary 3.3. For all $v=0,1,2,\ldots,$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}}K_\lambda(K_\mu(f_1,f_2),f_3)-K_\lambda(f_1,K_\mu(f_2,f_3))=0. \tag{3.24}$$

The interesting feature is that the $\circ$ product induces a deformation of the Poisson bracket Lie algebra as well. Indeed, we can define the bracket

$$\{\{f_1,f_2\}\}:=-(\mathrm{i}/\hbar)(f_1\circ f_2-f_2\circ f_1) \tag{3.25}$$

which satisfies

$$\{\{f_1\circ f_2,f_3\}\}=f_1\circ\{\{f_2,f_3\}\}+\{\{f_1,f_3\}\}\circ f_2. \tag{3.26}$$

Theorem 3.4. The space of classical physical observables, endowed with the $\{\{\ ,\ \}\}$ bracket, is a Lie algebra which is isomorphic to the quantum algebra of the corresponding operators under the $-(\mathrm{i}/\hbar)[\ ,\ ]$ commutator.

The proof is an immediate consequence of the definitions of the relevant algebras.

The expansion (3.20) generates an $\hbar$ expansion of the $\{\{f_1,f_2\}\}$ bracket in the form

$$\{\{f_1,f_2\}\}=\frac{1}{(2\pi)^n}\int \mathrm{d}^n\alpha\,\mathrm{d}^n\beta\,\tilde{f}_1(\alpha)\tilde{f}_2(\beta)\sum_{k=1}^{n}\left.\frac{\mathrm{d}(\alpha\circ'\beta-\beta\circ'\alpha)_k}{\mathrm{d}\hbar}\right|_{\hbar=0}P_{A_k}(s)$$
$$\times\exp\left(\mathrm{i}\sum_{l=1}^{n}(\alpha+\beta)_l P_{A_l}(s)\right)+\mathrm{O}(\hbar) \tag{3.27}$$

whose $\mathrm{O}(\hbar)$ terms are determined by the $\circ'$ group law. For theories with linear phase spaces one again recovers Moyal's results. This will be demonstrated in § 4, together with the $Q=\mathbb{R}_+$ example.

Proposition 3.5. In the $\hbar \to 0$ limit, the $\{\{f_1, f_2\}\}$ bracket contracts to the Poisson bracket for all polynomial functions of the basic observables.

Proof. First notice that $\{\{p_{A_l}, p_{A_j}\}\} = \{p_{A_l}, p_{A_j}\}_{\rm PB}$ for all the basic observables, as $\vartheta(p_{A_l}) = \hat{A}_l$. For monomial functions of the basic observables, equations (3.22), (3.26) and (3.27) imply

$$\begin{aligned}\{\{P_{A_l} \cdot P_{A_j}, P_{A_k}\}\} &= \{\{P_{A_l} \circ P_{A_j}, P_{A_k}\}\} + \mathrm{O}(\hbar) \\ &= P_{A_l} \circ \{P_{A_j}, P_{A_k}\}_{\rm PB} + \{P_{A_l}, P_{A_k}\}_{\rm PB} \circ P_{A_j} + \mathrm{O}(\hbar) \\ &= \{P_{A_l} \cdot P_{A_j}, P_{A_k}\}_{\rm PB} + \mathrm{O}'(\hbar).\end{aligned}$$

Similarly, we generalise to any polynomial function of the basic obervables and hence

$$\{\{f_1, f_2\}\} = \{f_1, f_2\}_{\rm PB} + \mathrm{O}(\hbar)$$

as stated. For non-polynomial functions of the basic observables the Poisson bracket limit can be ensured by requiring that, in the limit $\hbar \to 0$, the $\{\{\ ,\ \}\}$ bracket becomes a local Lie algebra† as defined by Kirillov (1976a, b). This local property distinguishes the Poisson algebra among all possible Lie algebras that can be defined on $C^\infty(S, \mathbb{R})$. A necessary and sufficient condition for the $\{\{\ ,\ \}\}$ bracket in equation (3.25) to become a local Lie algebra as $\hbar \to 0$ is that the $\hbar$ deformations of G are such that $(\mathrm{d}/\mathrm{d}\hbar)(\alpha \circ' \beta)_{l|\hbar=0}$ are linear in both α_l and β_l.

In particular, provided that the $L(\mathrm{G}^\hbar)$ commutation relations are of the following form:

$$[\hat{A}_l, \hat{A}_j] = \mathrm{i}\hbar C^k_{lj} \hat{A}_k$$

C^k_{lj} being the structure constants of $L(\mathrm{G})$, the Baker–Hausdorff formulae

$$\exp\left(\mathrm{i} \sum_{l=1}^n \alpha_l \hat{A}_l\right) \exp\left(\mathrm{i} \sum_{l=1}^n \beta_l \hat{A}_l\right) = \exp\left(\mathrm{i} \sum_{l=1}^n (\alpha+\beta)_l \hat{A}_l - \frac{\mathrm{i}\hbar}{2} \sum_{l,j,k=1}^n \alpha_l \beta_j C^k_{lj} \hat{A}_k + \mathrm{O}(\hbar^2)\right)$$

imply that

$$\left.\frac{\mathrm{d}(\alpha \circ' \beta)_k}{\mathrm{d}\hbar}\right|_{\hbar=0} = -\tfrac{1}{2} \sum_{l,j=1}^n \alpha_l \beta_j C^k_{lj} = -\left.\frac{\mathrm{d}(\beta \circ' \alpha)_k}{\mathrm{d}\hbar}\right|_{\hbar=0}.$$

In summary, what we have constructed here is a deformation of the Poisson Lie algebra,

$$\{\{f_1, f_2\}\} = \{f_1, f_2\}_{\rm PB} + \hbar \Lambda_1(f_1, f_2) + \hbar^2 \Lambda_2(f_1, f_2) + \ldots \tag{3.28}$$

where $\Lambda_1, \Lambda_2, \ldots$ are specified by the G-group product law. This coincides with the general results of Bayen *et al.* Here it has been derived using different methods, i.e. through explicit correspondence rules rather than general mathematical considerations of deformations of the Poisson algebra.

4. Examples

The generalised Lie bracket $\{\{\ ,\ \}\}$ algebra that has been discussed above may be quantised unambiguously via $\{\{\ ,\ \}\} \to -(\mathrm{i}/\hbar)[\ ,\ ]$ which thus replaces the Dirac

† $[\ ,\ ]$ is a local Lie algebra over $C^\infty(M, \mathbb{R})$ if (i) it is a Lie algebra, (ii) $[s_1, s_2]$ is continuous jointly in the variables s_1, s_2 and (iii) the support $\mathrm{supp}[s_1, s_2] \subset \mathrm{supp}\, s_1 \cap \mathrm{supp}\, s_2$ for all s_1, s_2 in $C^\infty(M, \mathbb{R})$.

quantisation algorithm $\{\ ,\ \}_{\mathrm{PB}} \to -(\mathrm{i}/\hbar)[\ ,\ ]$. As such it provides an intermediate (although unphysical) classical structure best suited for describing the relation between the classical and quantum theories. The general expression for the $\{\{\ ,\ \}\}$ bracket has been constructed in terms of the canonical group product law (cf (3.27)). Its study involves the use of the whole of $L^*(\mathrm{G})$ rather than just an individual Kirillov orbit which is locally isomorphic to a given G-homogeneous phase space S.

The simplest examples that may be demonstrated are provided by the theories with classical configuration spaces $Q = \mathbb{R}, \mathbb{R}_+$ whose associated canonical groups are G_{W} (Heisenberg-Weyl group) and $\mathbb{R}Ⓢ\mathbb{R}_+$ (affine group), respectively (Isham 1984).

4.1. $(Q = \mathbb{R})$

Following the group theoretical approach to quantisation that has been presented in § 2, we consider the translation group in two dimensions $T(2) = \mathbb{R}\times\mathbb{R}$ which acts transitively, effectively and symplectically on the phase space $T^*\mathbb{R} = \mathbb{R}\times\mathbb{R}$ by means of

$$l_{(u,v)}(q, p) := (q+u, p-v) \tag{4.1}$$

with corresponding vector fields $\gamma^{(a,b)}((a, b) \in L(T(2)))$

$$\gamma^{(a,b)} = -a\,\partial/\partial q + b\,\partial/\partial p. \tag{4.2}$$

The associated momentum map

$$P^{(a,b)}(s) = ap(s) + bq(s) \tag{4.3}$$

$s \in \mathbb{R}\times\mathbb{R}$, yields

$$\{P^{(a_1b_1)}, P^{(a_2b_2)}\}_{\mathrm{PB}} = b_1a_2 - b_2a_1 \tag{4.4}$$

which corresponds to a non-trivial cocycle of $\mathbb{R}\times\mathbb{R}$. Employing a central extension, with the aid of $\mathbb{R}$ we arrive at the Heisenberg-Weyl group $\mathrm{G_W}$ as the canonical group with the product law

$$(u_1, v_1, t_1)(u_2, v_2, t_2) = (u_1+u_2, v_1+v_2, t_1+t_2+\tfrac{1}{2}(v_1u_2 - v_2u_1). \tag{4.5}$$

The non-trivial Kirillov orbits in $L^*(\mathrm{G_W})$ are found to be 2-planes perpendicular to the x_3 axis. With the following parametrisation for the $\mathrm{G_W}$ group,

$$\hat{U}(\alpha_1, \alpha_2, \alpha_3) = \exp[\mathrm{i}(\alpha_1\hat{q} + \alpha_2\hat{p} + \hat{\alpha_2})] \tag{4.6}$$

the correspondence rule (3.11) is

$$\hat{F} = \frac{1}{(2\pi)^3}\int \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\tilde{f}(\alpha, \beta, \gamma)\,\hat{U}(\alpha, \beta, \gamma) \tag{4.7}$$

with

$$\tilde{f}(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^3}\int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,f(x_1, x_2, x_3)\exp[-\mathrm{i}(\alpha x_1 + \beta x_2 + \gamma x_3)] \tag{4.8}$$

where $f(x_1, x_2, x_3)$ is the extension of a classical observable $f(q, p)$ on a Kirillov orbit to the whole of $L^*(\mathrm{G_W})$ by replacing q with x_1 and p with x_2 according to the general procedure described in § 3. In this case, the x_3 and γ_3 integrations can be trivially performed and the usual Weyl correspondence rule results (cf Weyl 1928). One easily obtains (cf (3.27))

$$\{\{f_1, f_2\}\} = \{f_1, f_2\}_{\mathrm{PB}} + \sum_{k=1}^{\infty}\frac{1}{(2k+1)!}(\tfrac{1}{2}\mathrm{i}\hbar)^{2k} f_1 \overleftrightarrow{P}^{2k+1} f_2 \tag{4.9}$$

with (summation over l and j is assumed)

$$\overleftrightarrow{P}^{2k+1} := \frac{\overleftarrow{\partial}^{2k+1}}{\partial s^{l_1} \ldots \partial s^{l_{2k+1}}} c^{l_1 j_1} \ldots c^{l_{2k+1} j_{2k+1}} \frac{\overrightarrow{\partial}^{2k+1}}{\partial s^{j_1} \ldots \partial s^{j_{2k+1}}} \tag{4.10}$$

where c^{lj} are the components of the inverse of the symplectic form ω in the $s^l = (q, p)$ coordinate system. The bracket (4.9) coincides with the Moyal bracket for $Q = \mathbb{R}$ (Moyal 1949).

4.2. ($Q = \mathbb{R}_+$)

The group $\mathbb{R} \circledS \mathbb{R}_+$ acts transitively, effectively and symplectically on $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}_+$ by

$$l_{(v,\lambda)}(q, p) = (\lambda q, \lambda^{-1} p - v) \tag{4.11}$$

with corresponding Hamiltonian vector fields

$$\gamma^{(b,r)} = -rq \frac{\partial}{\partial q} + (b + rp) \frac{\partial}{\partial p} \tag{4.12}$$

where $(b, r) \in L(\mathbb{R} \circledS \mathbb{R}_+)$. The associated map

$$P^{(b,r)}(s) = bq(s) + rq(s)p(s) \tag{4.13}$$

$s \in \mathbb{R} \times \mathbb{R}_+$ yields

$$\{P^{(b_1,r_1)}, P^{(b_2,r_2)}\}_{\mathrm{PB}} = P^{(b_1 r_2 - b_2 r_1, 0)} \tag{4.14}$$

i.e. there are no cocycle obstructions and so $\{q, qp := \pi\}$ are the basic observables to base quantisation on $\mathbb{R}_+$ with commutation relations

$$[\hat{q}, \hat{\pi}] = \mathrm{i}\hbar\hat{q}. \tag{4.15}$$

The non-trivial Kirillov orbits in $L^*(\mathbb{R} \circledS \mathbb{R}_+)$ are two 2-half planes, each of which ($x_1 > 0$ or $x_1 < 0$) is globally diffeomorphic to the phase space $\mathbb{R} \times \mathbb{R}_+$. The correspondence rule (3.11) is obtained by extending any classical physical observable $f(q, \pi)$ to $f(x_1, x_2)$ defined on the whole of $L^*(\mathbb{R} \circledS \mathbb{R}_+)$ by simply replacing q with x_1 and π with x_2. With the parametrisation

$$\hat{U}(\alpha_1, \alpha_2) = \exp[\mathrm{i}(\alpha_1 \hat{q} + \alpha_2 \hat{\pi})] \tag{4.16}$$

for the canonical group $\mathbb{R} \circledS R_+$, the product law is

$$(\alpha_1, \alpha_2) \circ' (\beta_1, \beta_2)$$
$$= \left(\frac{\alpha_2 + \beta_2}{1 - \exp[\hbar(\alpha_2 + \beta_2)]} \left[\frac{\alpha_1}{\alpha_2} \left(1 - \exp(\hbar\alpha_2) + \frac{\beta_1}{\beta_2} \exp(\hbar\alpha_2)(1 - \exp(\hbar\beta_2)) \right) \right], \alpha_2 + \beta_2 \right) \tag{4.17}$$

which induces an $\hbar$ expansion of the $\{\{\ ,\ \}\}$ bracket (cf (3.27)) given by

$$\{\{f_1, f_2\}\} = \{f_1, f_2\}_{\mathrm{PB}} + \hbar^2 \Lambda_2(f_1, f_2) + \mathrm{O}(\hbar^3) \tag{4.18}$$

where

$$\Lambda_2(f_1,f_2)=\tfrac{1}{24}q^3\left(\frac{\partial^3 f_1}{\partial\pi^3}\frac{\partial^3 f_2}{\partial q^3}-\frac{\partial^3 f_2}{\partial\pi^3}\frac{\partial^3 f_1}{\partial q^3}-3\frac{\partial^3 f_1}{\partial q\,\partial\pi^2}\frac{\partial^3 f_2}{\partial q^2\,\partial\pi}+3\frac{\partial^3 f_1}{\partial q^2\,\partial\pi}\frac{\partial^3 f_2}{\partial q\,\partial\pi^2}\right)$$
$$+\tfrac{1}{2}q^2\left(\frac{\partial^3 f_1}{\partial\pi^3}\frac{\partial^2 f_2}{\partial q^2}-\frac{\partial^2 f_1}{\partial q^2}\frac{\partial^3 f_2}{\partial\pi^3}+\frac{\partial^3 f_1}{\partial q^2\,\partial\pi}\frac{\partial^2 f_2}{\partial\pi^2}\right.$$
$$\left.-\frac{\partial^2 f_1}{\partial\pi^2}\frac{\partial^3 f_2}{\partial q^2\,\partial\pi}+2\frac{\partial^2 f_1}{\partial q\,\partial\pi}\frac{\partial^3 f_2}{\partial q\,\partial\pi^2}-2\frac{\partial^3 f_1}{\partial q\,\partial\pi^2}\frac{\partial^2 f_2}{\partial q\,\partial\pi}\right)$$
$$+\tfrac{1}{6}q\left(\frac{\partial^2 f_1}{\partial q\,\partial\pi}\frac{\partial^2 f_2}{\partial\pi^2}-\frac{\partial^2 f_1}{\partial\pi^2}\frac{\partial^2 f_2}{\partial q\,\partial\pi}\right). \tag{4.19}$$

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